The birth of a cut in unitary random matrix ensembles

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Outline

1. Unitary random matrix ensembles
2. Critical ensembles
3. The birth of a cut
4. Riemann-Hilbert problems

Some well-known facts (2):

- Eigenvalues of a random matrix follow a determinantal point process with correlation kernel

$$K_n(x,y) := e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \sum_{k=0}^{n-1} p_k(x)\, p_k(y),$$

  p_k orthonormal polynomials w.r.t. weight $e^{-nV(x)}$ on $\mathbb{R}$

- Kernel contains information about eigenvalues
  - m-point correlation functions,
  - largest eigenvalue distribution,
  - gap probabilities, ...

1. Unitary random matrix ensembles

- We are interested in local behavior of eigenvalues near some reference point x^*
  - local scaling limits of the kernel

$$\lim_{n\to\infty} \frac{1}{cn^\beta} K_n\left(x^* + \frac{u}{cn^\beta}, x^* + \frac{v}{cn^\beta}\right)$$

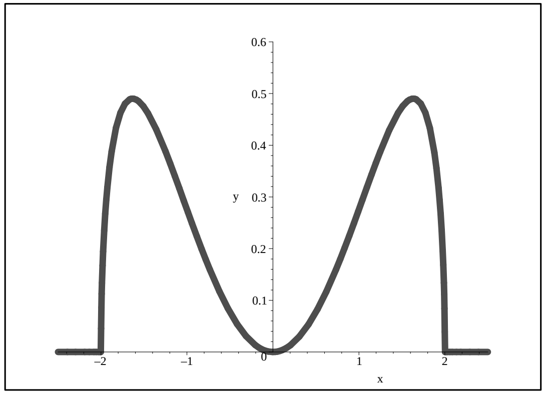
- in the bulk of the spectrum: sine kernel for $\beta = 1$, (Dyson, Deift-Kriechterbauer-McLaughlin-Venakides-Zhou, Bleher-Its, Pastur-Shcherbina)

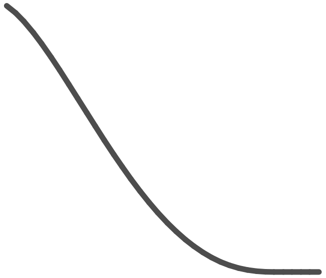
- at the edge of the spectrum (generically): Airy kernel for $\beta = 2/3$ (Forrester, Tracy-Widom, DKMVZ, BI, Deift-Gioev)

$\longrightarrow$ **Universality**

2. Critical ensembles

- Universality breaks down in three cases:





2. Critical ensembles

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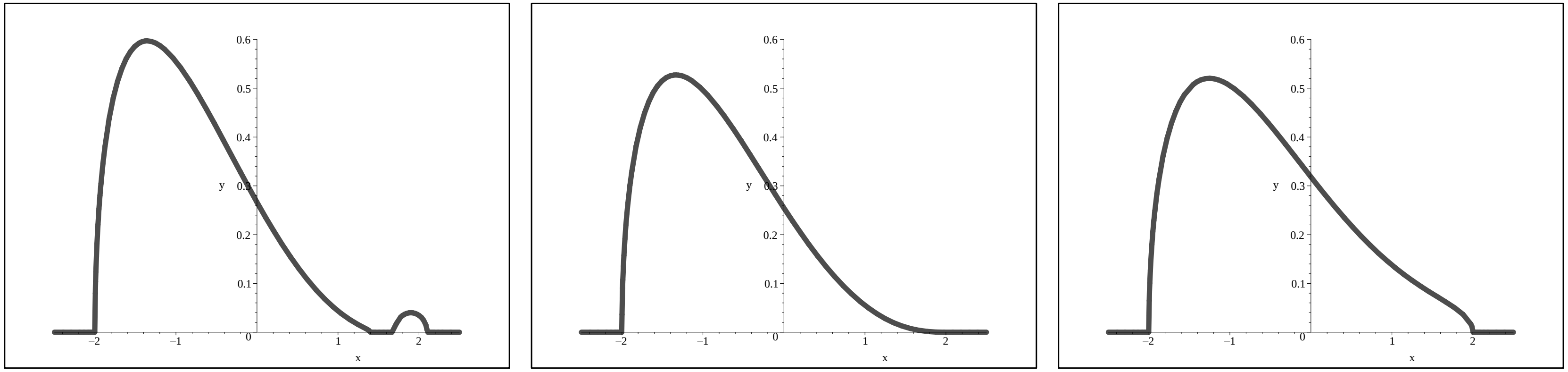
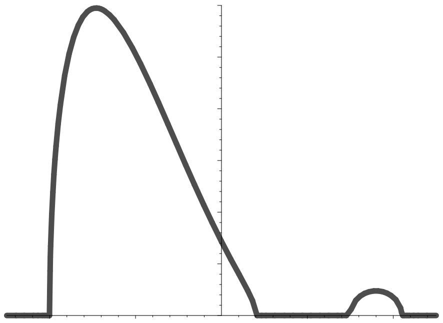
- Critical ensembles indicate a possible change of the number of intervals in the support
  - two merging intervals, with one of them shrinking at the same time



2. Critical ensembles

- Critical ensembles indicate a possible change of the number of intervals in the support
  - birth of a cut away from the spectrum - disappearing of an interval



2. Critical ensembles

- Including a parameter in the potential, $V \to V_t$, leads to those transitions
- Local eigenvalue behavior in the transitions is described by double scaling limits of the eigenvalue correlation kernel
  - $n \to \infty$ and $t \to t_c$ at an appropriate rate
- no sine or Airy kernel in critical cases,
  -

The birth of a cut

- studied in physics literature by Eynard (2006)
- mathematics literature:
  ’independent and simultaneous’ works by
  Mo, Bertola-Lee, and myself
  (3 papers appeared on arXiv 19-26 november 2007)

3. The birth of a cut

- We assume a potential V such that supp ρ_V a, b , with a singular exterior point $x^* > b$,

3. The birth of a cut

- limiting kernel in the birth of a cut-transition?
- double scaling limit where we let $n \to \infty$ and at the same time $t \to 1$
  - appropriate rate of convergence turns out to be such that $t - 1 = \mathcal{O}\left(\frac{\log n}{n}\right)$
  - bounded number of eigenvalues expected in the new cut
- We write

$$\nu = c_V(t-1)\frac{n}{\log n}$$

and let $n \to \infty$, $t \to 1$ in such a way that

$$\nu \to \nu_0.$$

3. The birth of a cut

The result:

- In the double scaling limit, we have

$$\lim \frac{1}{(cn)^{1/2}} K_{n,t}\left(x^* + \frac{u}{(cn)^{1/2}}, x^* + \frac{v}{(cn)^{1/2}}\right) = \begin{cases} \mathbb{K}^{\mathrm{GUE}}(u,v;k) & \text{for } k-\frac{1}{2} < \nu_0 < k+\frac{1}{2},\ k \geq 1, \\ 0 & \text{for } \nu_0 < 1/2, \end{cases}$$

$$\mathbb{K}^{\mathrm{GUE}}(u,v;k) = \frac{e^{-\frac{u^2+v^2}{2}}}{2^k\sqrt{\pi}(k-1)!} \frac{H_k(u)H_{k-1}(v) - H_k(v)H_{k-1}(u)}{u-v},$$

where H_k are Hermite polynomials

3. The birth of a cut

- $t < 1$, $\nu_0 < 0$: no eigenvalues expected, trivial limiting kernel
- when ν_0 increases, more eigenvalues 'move' to the new cut
- eigenvalues in the new cut seem to behave like the eigenvalues in a finite GUE
- Discontinuity of limiting kernel when ν_0 is a half integer?

3. The birth of a cut

$$\frac{1}{(cn)^{1/2}} K_{n,t}\left(x^* + \frac{u}{(cn)^{1/2}}, x^* + \frac{v}{(cn)^{1/2}}\right)$$

$$\lambda^{-}_{n,t}\mathbb{K}^{\mathrm{GUE}}(u, v; k)$$

$$+\lambda^{+}_{n,t}\mathbb{K}^{\mathrm{GUE}}(u, v; k + 1) + \mathcal{O}\left(\frac{\log n}{n^{1}}\right)$$

4. Riemann-Hilbert problems

- Goal is to find asymptotics for Y in the double scaling limit
- Deift/Zhou steepest descent method
  - series of transformations, 'undressing' of the RH problem
  - $Y \mapsto T \mapsto S \mapsto R$
  - $R(z$ $I + o(1$ $\quad \Rightarrow \quad$ asymptotics for Y
  - Two crucial features
    - $\rightarrow$ Construction of g-function using modified equilibrium measures
    - $\rightarrow$ Construction of local parametrix near x^* using RH problem for Hermite polynomials