

# Random matrices with equi-spaced external source

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## Random matrices with external source

- space of  $n \times n$  Hermitian matrices with probability measure

$$\overline{\mathbf{Z}}_n \propto \int \exp(-n \operatorname{Tr} (\mathbf{V}(\mathbf{M}) - \mathbf{A}\mathbf{M})) d\mathbf{M},$$

where

- ▶  $\mathbf{V}$  is a polynomial of even degree with positive leading

- if  $A$  , unitary ensemble

$$\overline{Z}_n = \int \exp(-n \text{Tr} V(M)) dM.$$

- we will study the case

$$A = \text{diag}(1, \dots, n)$$

- ▶ for  $V(x) = cx^2$ , eigenvalues behave like  $n$  non-intersecting Brownian motions starting at 0 and ending at  $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$

# Random matrices with external source

- Joint probability distribution of eigenvalues in the ensemble

$$\frac{1}{Z_n} \int \exp(-n \text{Tr}(V(M) - AM)) dM$$

is given by

$$\frac{1}{Z_n} \int \prod_{i,j=1,\dots,n} \prod_{i < j} (e^{na_i \lambda_j} - e^{na_j \lambda_i}) \prod_{j=1}^n e^{-nV(\lambda_j)} d\lambda_j$$

► if  $A = \frac{1}{n} \text{diag}(a_1, \dots, a_n)$ ,

$$\frac{1}{Z_n} \int \prod_{i < j} (e^{\lambda_i} - e^{\lambda_j}) \prod_{j=1}^n e^{-nV(\lambda_j)} d\lambda_j.$$

- Behavior of eigenvalues for large  $n$ ?

- $\mathbf{A} = \text{diag}(\mathbf{a}, \dots, \mathbf{a}, -\mathbf{a}, \dots, -\mathbf{a})$  (*Bleher-Kuijlaars, Bleher-Delvaux-Kuijlaars, Adler-van Moerbeke*)
  - ▶ vector equilibrium problem
  - ▶ critical point: Pearcey kernel
- $\mathbf{A} = \text{diag}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k, \dots, \mathbf{a}_k)$  with  $k$  fixed (*Baik-Wang, Bertola-Buckingham-Lee-Pierce, Adler-Délépine-van Moerbeke*)
  - ▶ every non-zero eigenvalue of  $\mathbf{A}$  is responsible for at most one outlier-eigenvalue of  $\mathbf{M}$
- External source matrix with  $n$  different eigenvalues (*Eynard-Orantin*)

# External source

■ **A**  $\frac{1}{n} \text{diag} \left( \frac{1}{z_n}, \dots, \frac{1}{z_n}, n - \frac{1}{z_n} \right)$  pdf  
 $\frac{1}{z_n} \prod_{i < j} (\lambda_i - \lambda_j) \prod_{i < j} (e^{\lambda_i} - e^{\lambda_j}) \prod_{j=1}^n e^{-nV(\lambda_j)} d\lambda_j$

■ eigenvalue configurations for which

$$\frac{1}{2n} \sum_{i < j} \log(\lambda_i - \lambda_j)^{-1} + \frac{1}{2n} \sum_{i < j} \log(e^{\lambda_i} - e^{\lambda_j}) \log \lambda_j$$

# Limiting mean eigenvalue density

- limiting mean distribution minimizes

$$-\int \int \log|t - s|^{-1} d\mu(t) d\mu(s) \quad - \int \log|e$$





# Eigenvalue correlation kernel

Random matrices with external source

$\mathbf{A} = \frac{1}{n} \text{diag}( \dots, n-1, n-1 )$

- correlation kernel for eigenvalues is given by

$$\mathbf{K}_n(\mathbf{x}, \mathbf{y}) = \int_{-\infty}^{\infty} e^{-\frac{n}{2} V(x)} e^{-\frac{n}{2} V(y)} \sum_{k=0}^{n-1} \mathbf{p}_k(\mathbf{x}) \mathbf{q}_k(\mathbf{y}) dx$$

- polynomials  $\mathbf{p}_k$  of degree  $k$  and  $\mathbf{q}_j$  of degree  $j$  are determined by the orthogonality conditions

$$\int_{-\infty}^{\infty} \mathbf{p}_k(\mathbf{x}) \mathbf{q}_j(\mathbf{x}) e^{-nV(x)} dx = \delta_{kj}$$

- $\mathbf{p}_k$ 's are type II multiple OPs with  $n$  orthogonality weights

$$1, e^x, e^{2x}, \dots, e^{(n-1)x}$$

# Eigenvalue correlation kernel

Random matrices with external source

$\mathbf{A} = \frac{1}{n} \text{diag}(a_1, \dots, a_{n-1}, a_n)$  :

- correlation kernel for eigenvalues is given by

$$\mathbf{K}_n(\mathbf{x}, \mathbf{y}) = \int_{-1}^{n-1} e^{-\frac{n}{2} V(x)} e^{-\frac{n}{2} V(y)} \mathbf{p}(\mathbf{x}) \mathbf{q}(y) e^{-\frac{n}{2} V(x)} dx$$

- ▶ polynomials  $\mathbf{p}_k$  of degree  $k$  and  $\mathbf{q}_j$  of degree  $j$  are determined by the orthogonality conditions

$$\int_{-1}^{n-1} \mathbf{p}_k(x) \mathbf{q}_j(x) e^{-nV(x)} dx = \delta_{kj}$$

- ▶  $\mathbf{q}_j$ 's are related to type I multiple orthogonal polynomials

# Eigenvalue correlation kernel

Interpretation of the polynomials in terms of the random matrix ensemble

$$\overline{Z}_n = \int \exp(-n \text{Tr}(V(M) - \text{AM})) dM$$

or the determinantal point process

$$\overline{Z}_n = \int \prod_{i < j} (e^{\lambda_i} - e^{\lambda_j}) \prod_{j=1}^n e^{-nV(\lambda_j)} d\lambda_j.$$

■ p



# Eigenvalue correlation kernel

# Eigenvalue correlation kernel

## ■ RH problem for usual OPs (*Fokas-Its-Kitaev '92*)

(a)  $Y$  is analytic in  $\mathbb{C} \setminus \mathcal{R}$ ,

(b)  $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$  for  $x \in \mathcal{R}$ ,

(c)  $Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$  as  $z \rightarrow \infty$ ,

## ■ Unique solution given by

$$Y(z) = \begin{pmatrix} \kappa_n^- p_n(z) - \frac{i}{n} \int_{\mathcal{R}} \frac{p_n(s) w(s)}{s-z} ds & \\ -2i\kappa_{n-1}^- p_{n-1}(z) - \frac{i}{n-1} \int_{\mathcal{R}} \frac{p_{n-1}(s) w(s)}{s-z} ds & \end{pmatrix},$$

# Eigenvalue correlation kernel

- polynomials defined by

$$p_n(x) = \int_{\mathbb{R}} q(x) (e^x - e^{-n\psi(x)}) dx$$

- standard RH problem for MOPs is of size  $n$  - inconvenient for  $n$  large

- let

$$Y_1(z) = \begin{pmatrix} 1 & \\ & p_n(z) \end{pmatrix}$$

and

$$Y_2(z) = \frac{1}{i} \int_{\mathbb{R}} \frac{p_n(s)}{e^{-s} - e^s} e^{-n\psi(s)} ds.$$



# RH problem for polynomials

1.  $Y = (Y_1, Y_2)$ , where  $Y_1$  is analytic in  $\mathbb{C} \setminus \mathbb{R}$ , and  $Y_2$  is analytic in



# RH problem for polynomials

- there is also a  $n \times n$  matrix RH problem
  - ▶ unlike for usual orthogonal polynomials,  $t Y(z)$
  - ▶ taking inverses is not possible
  - ▶ no advantage

- there is a dual RH problem for  $Y = (Y_1, Y_2)$ , where

$$Y_1 = \begin{pmatrix} q_n(e^s) \\ \vdots \\ q_1(e^s) \end{pmatrix}, \quad Y_2(z) = \frac{1}{i} \int_{\gamma} \frac{q_n(e^s)}{z-s} e^{-n\psi(s)} ds.$$

# RH problem for polynomials

1.  $Y = (Y_1, Y_2)$ , where  $Y_2$

# RH problem for polynomials

- Asymptotic analysis of the RH problem if the support of  $\mu$  is one interval: Deift/Zhou steepest descent analysis

- Modifications compared to analysis for OPs

- ▶ construction of two  $g$ -functions

$$g(z) \sim \int \log(z - y) d\mu(y)$$

$$g(z) \sim \int \log(e - e^{-y}) d\mu(y).$$

- ▶ Crucial step: transformation of the RH problem to a non-local scalar RH problem in the complex plane

# RH problem for polynomials

- Transformation to shifted RH problem of the form

1.  $F \in \mathbb{C} \setminus \mathbb{C}$  is analytic

2. for  $z \in \mathbb{C}$ , we have

$$F_+(z) - F_-(z) = J_n(z) F_{\pm}(f(z) - J_n(z)),$$

with  $f \in \mathbb{C}$ ,

3.

# Outlook

- Universality

- ▶ sine kernel
- ▶ Airy kernel

- multi-cut case

- large  $n$  behavior in more general point processes of the form

$$\overline{\mathbf{z}}_n = \prod_{i < j} (x_i - x_j) \prod_{i < j} (f(x_i) - f(x_j)) \prod_{j=1}^r e^{-nV(\lambda_j)} \mathbf{d}_j.$$